Uniformly Constructing a Series of Nonlinear Wave and Coefficient Functions' Soliton Solutions and Double Periodic Solutions for the (2 + 1)-Dimensional Broer-Kaup-Kupershmidt Equation

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By using a new more general ansatz with the aid of symbolic computation, we extended the unified algebraic method proposed by Fan [Computer Phys. Commun. **153**, 17 (2003)] and the improved extended tanh method by Yomba [Chaos, Solitons and Fractals **20**, 1135 (2004)] to uniformly construct a series of soliton-like solutions and double-like periodic solutions for nonlinear partial differential equations. The efficiency of the method is demonstrated on the $(2+1)$-dimensional Broer-Kaup-Kupershmidt equation.

Key words: Generalized Algebraic Method; Symbolic Computation; Solitary Wave Solution; Weierstrass and Jacobi Elliptic Functions; Periodic Solution.

1. Introduction

The tanh method [1 – 3] provides a straightforward and effective algorithm to obtain particular solutions for a large number of nonlinear equations. Recently, much research has been concentrated on various extensions and applications of the tanh method [1 – 6], because the availability of computer systems like Maple or Mathematica allow to perform some complicated and tedious algebraic and differential calculations on a computer.

Generally speaking, the various extensions and improvements of the tanh method can be classified into two classes: One is called the direct method, which represents the solutions of a given nonlinear partial differential equation (NPDE) as the sum of a polynomial in fundamental function solutions such as the tanh function [1 – 3], the hyperbolic-functions [4 – 6], the Jacobi elliptic functions expansion [7 – 10] and so on. It requires solving the recurrent relation or derivative relation for the terms of a polynomial. The more general the ansatz, the more general and more formal the solutions of the NPDEs will be. The second one is called the subequation method, which consists of looking for the solutions of a given NPDE as a polynomial in a variable which satisfies a certain subequation. For example, the Riccati equation [11, 12], the projective Riccati equation [13, 14], a degenerate or a nondegenerate elliptic equation [15 – 16], and so on.

In [17 – 18], Fan developed a new algebraic method, belonging to the subequation method, to seek new solitary wave solutions of NPDEs that can be expressed as a polynomal in an elementary function which satisfies a more general subequation than Riccati's equation [11, 12]. Compared with most of the existing tanh methods, the proposed method not only gives an unified formulation to construct various travelling wave solutions, but also provides a guideline to classify the various types of travelling wave solutions according to the values of some parameters. More recently, by means of a more general ansatz, Chen and Wang [19] further developed this method and constructed more solutions of NPDEs in terms of special functions. On the other hand, Yomba [20] uses an improved extended tanh method to obtain some new soliton-like solutions for the $(2+1)$-dimensional dispersive long wave equation. The present work is motivated by the intention to generalize the above work made in [9 – 11] by proposing a more general ansatz, so that it can be used to obtain more types and general formal solutions which contain not only the results obtained by using the method [17, 18] and the method [20] but also a series of nontravelling wave and coefficient functions' soliton-like solutions, double-like periodic solutions and triangular-like solutions for NPDEs, in which the restriction on $\xi(x,y,t)$ as merely a linear function x,y,t and the restrictions on the coefficients to be constants to be removed.

For illustration, we apply the generalized method to solve a $(2+1)$-dimensional Broer-Kaup-Kupershmidt (BKK) equation [21] and successfully construct new and more general solutions including a series of nontravelling wave and coefficient functions' soliton-like solutions, double-like periodic solutions and triangular-like solutions.

Our paper is organized as follows. In the following Section 2 the details of the derivation of the generalized algebraic method are given. The applications of the generalized method to the $(2+1)$-dimensional BKK equation are illustrated in Section 3. The conclusion is then given in the final Section 4.

2. Summary of the Generalized Method

In the following we would like to outline the main steps of our general method:

Step 1. Given a NPDE system with some physical fields $u_i(x,y,t)(i=1,\cdots,n)$ in three variables x, y, t,

$$F_i(u_i,u_{it},u_{ix},u_{iy},u_{itt},u_{ixt},u_{iyt},u_{ixx},u_{iyy},u_{ixy},\cdots)=0, \tag{2.1}$$

we express the solutions of the NPDE system by the new more general ansatz

$$u_i(\xi)=a_{i0}+\sum_{j=1}^{m_i}\left\{a_{ij}\phi^j+b_{ij}\phi^{j-1}\sqrt{\sum_{\rho=0}^{4}h_\rho\phi^\rho}\right\}, \tag{2.2}$$

where m_i is an integer to be determined by balancing the highest-order derivative terms with the nonlinear terms in (2.1), the new variable $\phi=\phi(\xi)$ satisfies:

$$\phi'=\frac{d\phi}{d\xi}=\sqrt{\sum_{\rho=0}^{4}h_\rho\phi^\rho}, \tag{2.3}$$

and $a_{i0}=a_{i0}(x,y,t)$, $a_{ij}=a_{ij}(x,y,t)$, $b_{ij}=b_{ij}(x,y,t)$ $(i=1,2,\cdots;\ j=1,2,\cdots,m_i)$, and $\xi=\xi(x,y,t)$ are all differentiable functions to be determined later. Here h_0, h_1, h_2, h_3, h_4 are constants.

Step 2. Substitute (2.2) into (2.1) along with (2.3), and then set all coefficients of $\phi^p\left(\sqrt{\sum_{\rho=0}^{r}h_\rho\phi^\rho}\right)^q$ $(q=0,1;\ p=0,1,2,\cdots)$ to be zero to get over-determined partial differential equations with respect to a_{i0}, a_{ij}, b_{ij} $(i=1,2,\cdots;\ j=1,2,\cdots,m_i)$ and ξ.

Step 3. Solving the over-determined partial differential equations by use of Maple, we would end up with explicit expressions for a_{i0}, a_{ij}, b_{ij} $(i=1,2,\cdots;\ j=1,2,\cdots,m_i)$ and ξ or the constrains among them.

Step 4. By using the results obtained in the above steps, we can derive a series of fundamental solutions such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions. Because we are interested in solitary waves, Jacobi and Weierstrass doubly periodic solutions, and tan and cot type solutions appearing in pairs with tanh and coth type solutions, respectively, we omit polynomial, rational, and triangular periodic solutions in this paper. By considering the different values of h_0, h_1, h_2, h_3 and h_4, (2.3) has many kinds of solitary-like wave, Jacobi and Weierstrass doubly periodic solutions which are listed as follows.

(i) Solitary wave solutions

a. Bell shaped soliton solutions

$$\phi=\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\xi),\qquad h_0=h_1=h_3=0,\quad h_2>0,\quad h_4<0, \tag{2.4}$$

$$\phi=-\frac{h_2}{h_3}\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right),\qquad h_0=h_1=h_4=0,\quad h_2>0. \tag{2.5}$$

b. Kink shaped soliton solutions

$$\phi=\sqrt{-\frac{h_2}{2h_4}}\tanh\left(\sqrt{-\frac{h_2}{2}}\xi\right),\qquad h_0=\frac{h_2{}^2}{4h_4},\quad h_1=h_3=0,\quad h_2<0,\quad h_4>0. \tag{2.6}$$

c. Soliton solutions

$$\phi=\frac{h_2\operatorname{sech}^2\left(\frac{1}{2}\sqrt{h_2}\xi\right)}{2\sqrt{h_2h_4}\tanh(\frac{1}{2}\sqrt{h_2}\xi)-h_3},\qquad h_0=h_1=0,\quad h_2>0. \tag{2.7}$$

(ii) Jacobi and Weierstrass doubly periodic solutions [22, 23]

$$\phi=\sqrt{\frac{-h_2m^2}{h_4(2m^2-1)}}\mathrm{cn}\left(\sqrt{\frac{h_2}{2m^2-1}}\xi\right),\ h_1=h_3=0,\quad h_4<0,\quad h_2>0,\quad h_0=\frac{h_2^2m^2(1-m^2)}{h_4(2m^2-1)^2},\tag{2.8}$$

$$\phi=\sqrt{\frac{-m^2}{h_4(2-m^2)}}\mathrm{dn}\left(\sqrt{\frac{h_2}{2-m^2}}\xi\right),\ h_1=h_3=0,\quad h_4<0,\quad h_2>0,\quad h_0=\frac{h_2^2(1-m^2)}{h_4(2-m^2)^2},\tag{2.9}$$

$$\phi=\sqrt{\frac{-h_2m^2}{h_4(m^2+1)}}\mathrm{sn}\left(\sqrt{-\frac{h_2}{m^2+1}}\xi\right),\ h_1=h_3=0,\quad h_4>0,\quad h_2<0,\quad h_0=\frac{h_2^2m^2}{h_4(m^2+1)^2},\tag{2.10}$$

$$\phi=\mathrm{ns}(\xi),\quad h_1=h_3=0,\quad h_4=1,\quad h_2=-(m^2+1),\quad h_0=m^2,\tag{2.11}$$

$$\phi=\mathrm{dc}(\xi),\quad h_1=h_3=0,\quad h_4=1,\quad h_2=-(m^2+1),\quad h_0=m^2,\tag{2.12}$$

$$\phi=\mathrm{nc}(\xi),\quad h_1=h_3=0,\quad h_4=1-m^2,\quad h_2=2m^2-1,\quad h_0=-m^2,\tag{2.13}$$

$$\phi=\mathrm{nd}(\xi),\quad h_1=h_3=0,\quad h_4=m^2-1,\quad h_2=2-m^2,\quad h_0=-1,\tag{2.14}$$

$$\phi=\mathrm{cs}(\xi),\quad h_1=h_3=0,\quad h_4=1,\quad h_2=2-m^2,\quad h_0=1-m^2,\tag{2.15}$$

$$\phi=\mathrm{sc}(\xi),\quad h_1=h_3=0,\quad h_4=1-m^2,\quad h_2=2-m^2,\quad h_0=1,\tag{2.16}$$

$$\phi=\mathrm{sd}(\xi),\quad h_1=h_3=0,\quad h_4=m^2(m^2-1),\quad h_2=2m^2-1,\quad h_0=1,\tag{2.17}$$

$$\phi=\mathrm{ds}(\xi),\quad h_1=h_3=0,\quad h_4=1,\quad h_2=2m^2-1,\quad h_0=m^2(m^2-1),\tag{2.18}$$

$$\phi=\mathrm{ns}(\xi)\pm\mathrm{cs}(\xi),\quad h_1=h_3=0,\quad h_4=\frac{1}{4},\quad h_2=\frac{1-2m^2}{2},\quad h_0=\frac{1}{4},\tag{2.19}$$

$$\phi=\mathrm{nc}(\xi)\pm\mathrm{sc}(\xi),\quad h_1=h_3=0,\quad h_4=\frac{1-m^2}{4},\quad h_2=\frac{1+m^2}{2},\quad h_0=\frac{1-m^2}{4},\tag{2.20}$$

$$\phi=\mathrm{ns}(\xi)\pm\mathrm{ds}(\xi),\quad h_1=h_3=0,\quad h_4=\frac{1}{4},\quad h_2=\frac{m^2-2}{2},\quad h_0=\frac{m^2}{4},\tag{2.21}$$

$$\phi=\mathrm{sn}(\xi)\pm\mathrm{icn}(\xi),\quad h_1=h_3=0,\quad h_4=\frac{m^2}{4},\quad h_2=\frac{m^2-2}{2},\quad h_0=\frac{m^2}{4},\tag{2.22}$$

where m is a modulus and $\mathrm{i}^2=-1$.

$$\phi=\wp\left(\frac{\sqrt{h_3}}{2}\xi,g_2,g_3\right),h_2=0,h_3>0,\tag{2.23}$$

where $g_2=-4\frac{h_1}{h_3}$ and $g_3=-4\frac{h_0}{h_3}$ are called invariants of the Weierstrass elliptic function. The Jacobi elliptic functions are doubly periodic and possess properties of triangular functions:

$$\mathrm{sn}^2\xi+\mathrm{cn}^2\xi=1,\quad \mathrm{dn}^2\xi=1-m^2\mathrm{sn}^2\xi,$$
$$(\mathrm{sn}\xi)'=\mathrm{cn}\xi\mathrm{dn}\xi,\quad (\mathrm{cn}\xi)'=-\mathrm{sn}\xi,$$
$$(\mathrm{dn}\xi)'=-m^2\mathrm{sn}\xi\mathrm{cn}\xi.$$

When $m\to1$, the Jacobi functions degenerate to hyperbolic functions, *i. e.*

$$\mathrm{sn}\xi\to\tanh\xi,\quad \mathrm{cn}\xi\to\mathrm{sech}\xi,$$

when $m\to0$, the Jacobi functions degenerate to triangular functions, *i. e.*

$$\mathrm{sn}\xi\to\sin\xi,\quad \mathrm{cn}\xi\to\cos\xi.$$

More detailed notations for the Weierstrass and Jacobi elliptic functions can be found in [21, 24].

Remarks:

1. Generalization

The method proposed here is more general than the method [17, 18] by Fan and the improved method [20] by Yomba. First, compared with the method [17, 18], the restriction on $\xi(x,y,t)$ as merely a linear function of x,y,t and the restriction on the coefficients a_{i0}, a_{ij}, b_{ij} ($i=1,2,\cdots;\ j=1,2,\cdots,m_i$) as constants are removed. Second, compared with the improved method [20] by Yomba, (2.3) for the new variable $\phi=\phi(\xi)$ is more general. More importantly, we add terms $b_{ij}\phi^{j-1}\sqrt{\sum_{\rho=0}^{4}h_\rho\phi^\rho}$ in our new ansatz (2.2), so more types of solutions would be expected for some equations.

2. Feasibility

Because of the generalization of the ansatz, a more complicated computation is expected than before. Even if the availability of computer symbolic systems like Maple or Mathematica allow us to perform the complicated and tedious algebraic calculations and differentiation on a computer, in general it is very difficult, sometime impossible, to solve the set of over-determined partial differential equations in *step 3*. As the calculation goes on, in order to drastically simplify the work or make the work feasible, we often choose special function forms for a_{i0}, a_{ij}, b_{ij} $(i=1,2,\cdots;\ j=1,2,\cdots,m_i)$ and ξ, on a trial-and-error basis.

3. Further extension

In fact, we naturally present a more general ansatz, which reads

$$u_i(\xi)=a_{i0}+\sum_{j=1}^{m_i}\left\{a_{ij}\phi^j+b_{ij}\phi^{-j}+f_{ij}\phi^{j-1}\sqrt{\sum_{\rho=0}^{r}h_\rho\phi^\rho}+k_{ij}\frac{\sqrt{\sum\limits_{\rho=0}^{r}h_\rho\phi^\rho}}{\varphi^j}\right\}, \tag{2.24}$$

where a_{i0}, a_{ij}, b_{ij}, f_{ij}, k_{ij} $(i=1,2,\cdots;\ j=1,2,\cdots,m_i)$ and ξ are differentiable functions to be determined later. We have studied in [19] the case where a_{i0}, a_{ij}, b_{ij}, f_{ij}, k_{ij} $(i=1,2,\cdots;\ j=1,2,\cdots,m_i)$ are constants and ξ is a linear function with respect to x, y and t in the above ansatze. Therefore, for some nonlinear equations, more types of solutions would be expected.

3. Exact Soliton-like Solutions of the (2 + 1)-Dimensional Broer-Kaup-Kupershmidt (BKK) Equation

Let us consider the BKK equations

$$\begin{aligned}&H_{ty}-H_{xxy}+2(HH_x)_y+2G_{xx}=0,\\&G_t+G_{xx}+2(HG)_x=0.\end{aligned} \tag{3.1}$$

The BKK system may be derived from the parameter dependent symmetry constraint of the Kadomtsev-Petviashvili (KP) equation [25]. Though the integrability of the BKK system can be guaranteed by the integrability of the KP equation (because it is a symmetry constraint of the KP equation), some authors have exactly proven its integrability in a different sense. For more details on the results of this system, the reader is advised to see the achievements in [25–31].

By balancing the highest-order contributions from both the linear and nonlinear terms in (3.1), we suppose that (3.1) has the following formal solutions:

$$H(x,y,t)=a_0+a_1\phi+b_1\sqrt{\sum_{\rho=0}^{4}h_\rho\phi^\rho},$$

$$\begin{aligned}G(x,y,t)=A_0+A_1\phi+B_1\sqrt{\sum_{\rho=0}^{4}h_\rho\phi^\rho}\\+A_2\phi^2+B_2\phi\sqrt{\sum_{\rho=0}^{4}h_\rho\phi^\rho},\end{aligned} \tag{3.2}$$

where $a_0=a_0(y,t)$, $a_1=a_1(y,t)$, $b_1=b_1(y,t)$, $A_0=A_0(y,t)$, $A_1=A_1(y,t)$, $B_1=B_1(y,t)$, $A_2=A_2(y,t)$, $B_2=B_2(y,t)$ and $\xi=kp+q(k=k(x),\ p=p(y,t)$ and $q=q(y,t))$ are all differential functions, and $\phi=\phi(\xi)$ satisfies (2.3).

With the aid of Maple, substituting (3.2) along with (2.3) into (3.1), yields a set of partial differential equations for $\phi^i\left(\sqrt{\sum_{\rho=0}^4 h_\rho\phi^\rho}\right)^j$, $(i=0,1,\cdots;\ j=0,1)$. Setting the coefficients of these terms $\phi^i\left(\sqrt{\sum_{\rho=0}^4 h_\rho\phi^\rho}\right)^j$ to zero yields a set of over-determined partial differential equations with respect to a_0, a_1, b_1, A_0, A_1, B_1, A_2, B_2, k, p, and q.

By use of Maple, solving the over-determined partial differential equations, we get

$$a_0=\frac{\pm C_3^2C_1^2\sqrt{h_4}h_3-2\left(\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)\right)h_4}{4C_1C_3h_4},$$

$$A_1=-\frac{1}{4}h_3\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1,$$

$$a_1=\pm\sqrt{h_4}C_3C_1,\quad p=C_3,$$

$$A_2=-\frac{1}{2}C_1C_3h_4\frac{\mathrm{d}}{\mathrm{d}y}F_2(y),$$

$$k=C_1x+C_2,\quad q=F_2(y)+F_1(t),$$

$$B_1 = \pm\frac{1}{2}\sqrt{h_4}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1, \quad A_0 = \pm\frac{\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1(4h_2h_4 - h_3^2)}{16h_4}, \quad b_1 = B_2 = 0. \tag{3.3}$$

From (3.2) and (3.3), we obtain the following families of solutions of (3.1).

Family 1. From (3.3), when $h_0 = h_1 = h_3 = 0$, $h_2 > 0$, and $h_4 < 0$, we obtain the following soliton solution for the BKK equation:

$$\begin{aligned} H_1 &= -\frac{1}{2}\frac{\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)}{C_1C_3} \pm \sqrt{h_4}C_3C_1\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}\left(\sqrt{h_2}\xi\right), \\ G_1 &= -\frac{1}{4}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1h_2 + \frac{1}{2}C_1C_3\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)h_2\operatorname{sech}^2\left(\sqrt{h_2}\xi\right) \\ &\quad \pm\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1\sqrt{-h_2^2\operatorname{sech}^2\left(\sqrt{h_2}\xi\right) + h_2^2\operatorname{sech}^4\left(\sqrt{h_2}\xi\right)} \end{aligned} \tag{3.4}$$

where $\xi = kp + q$, k, p and q are determined by (3.3).

Family 2. From (3.3), when $h_1 = h_3 = 0$, $h_0 = \frac{h_2^2}{4h_4}$, $h_2 < 0$, and $h_4 > 0$, we obtain the following soliton solution for the BKK equation:

$$\begin{aligned} H_2 &= -\frac{1}{2}\frac{\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)}{C_1C_3} \pm \frac{1}{2}\sqrt{h_4}C_3C_1\sqrt{-2\frac{h_2}{h_4}}\tanh\left(\frac{\sqrt{-2h_2}}{2}\xi\right), \\ G_2 &= \pm\frac{1}{4}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1h_2 + \frac{1}{4}C_1C_3\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)h_2\tanh^2\left(\frac{\sqrt{-2h_2}}{2}\xi\right) \\ &\quad \pm\frac{1}{4}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1\sqrt{h_2^2 - 2h_2^2\tanh^2\left(\frac{\sqrt{-2h_2}}{2}\xi\right) + h_2^2\tanh^4\left(\frac{\sqrt{-2h_2}}{2}\xi\right)}, \end{aligned} \tag{3.5}$$

where $\xi = kp + q$; k, p, and q are determined by (3.3).

Family 3. From (3.3), when $h_0 = h_1 = 0$, $h_2 > 0$ we obtain the soliton following solution for the BKK equation:

$$\begin{aligned} H_3 &= \frac{\pm C_3^2C_1^2\sqrt{h_4}h_3 - 2\left(\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)\right)h_4}{4C_1C_3h_4} \pm \frac{\sqrt{h_4}C_3C_1h_2\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)}{2\sqrt{h_2h_4}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3}, \\ G_3 &= \pm\frac{\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1(4h_2h_4 - h_3^2)}{16h_4} - \frac{h_3\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1h_2\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)}{4(2\sqrt{h_2h_4}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3)} \\ &\quad - \frac{C_1C_3h_4\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)h_2^2\operatorname{sech}\left(\frac{\sqrt{h_2}}{2}\xi\right)^4}{2\left(2\sqrt{h_2h_4}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)^2} \pm \frac{1}{2}\frac{\sqrt{h_4}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1}{\left(2\sqrt{h_2h_4}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)} \\ &\quad \cdot\sqrt{h_2^3\operatorname{sech}^4\left(\frac{\sqrt{h_2}}{2}\xi\right) + \frac{h_3h_2^3\operatorname{sech}^6\left(\frac{\sqrt{h_2}}{2}\xi\right)}{\left(2\sqrt{h_2h_4}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)^3} + \frac{h_4h_2^4\operatorname{sech}^8\left(\frac{\sqrt{h_2}}{2}\xi\right)}{\left(2\sqrt{h_2h_4}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)^4}}, \end{aligned} \tag{3.6}$$

where $\xi = kp + q$; k, p and q are determined by (3.3).

Family 4. From (3.3), when $h_1 = h_3 = 0$, $h_0 = \frac{h_2^2 m^2(1-m^2)}{h_4(2m^2-1)^2}$, $h_2 > 0$, and $h_4 < 0$, we obtain:

$$\begin{aligned} H_4 &= -\frac{1}{2}\frac{\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)}{C_1C_3} \pm \sqrt{h_4}C_3C_1\sqrt{-\frac{h_2m^2}{h_4(2m^2-1)}}\mathrm{cn}\left(\sqrt{\frac{h_2}{2m^2-1}}\xi\right), \\ G_4 &= \pm\frac{1}{4}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1h_2 + \frac{\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1h_2m^2\mathrm{cn}^2\left(\sqrt{\frac{h_2}{2m^2-1}}\xi\right)}{2(2m^2-1)} \\ &\quad \pm\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1\sqrt{\frac{h_2^2m^2(1-m^2)}{h_4(2m^2-1)^2} - \frac{h_2^2m^2\mathrm{cn}^2\left(\sqrt{\frac{h_2}{2m^2-1}}\xi\right)}{(2m^2-1)} + \frac{h_2^2m^4\mathrm{cn}^4\left(\sqrt{\frac{h_2}{2m^2-1}}\xi\right)}{(2m^2-1)^2}}, \end{aligned} \tag{3.7}$$

where $\xi = kp + q$; k, p, and q are determined by (3.3).

Family 5. From (3.3), when $h_1 = h_3 = 0$, $h_0 = \frac{h_2^2(1-m^2)}{h_4(2-m^2)^2}$, $h_2 > 0$, and $h_4 < 0$, we obtain the following solution for the BKK equation:

$$\begin{aligned} H_5 &= -\frac{1}{2}\frac{\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)}{C_1C_3} \pm \sqrt{h_4}C_3C_1\sqrt{-\frac{m^2}{h_4(2-m^2)}}\mathrm{dn}\left(\sqrt{\frac{h_2}{2-m^2}}\xi\right), \\ G_5 &= \pm\frac{1}{4}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1h_2 + \frac{\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1m^2\mathrm{dn}^2\left(\sqrt{\frac{h_2}{2-m^2}}\xi\right)}{2\,(2-m^2)} \\ &\quad \pm\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1\sqrt{\frac{h_2^2\,(1-m^2)}{(2-m^2)^2} - \frac{h_2m^2\mathrm{dn}^2\left(\sqrt{\frac{h_2}{2-m^2}}\xi\right)}{(2-m^2)} + \frac{m^4\mathrm{dn}^4\left(\sqrt{\frac{h_2}{2-m^2}}\xi\right)}{(2-m^2)^2}} \end{aligned} \tag{3.8}$$

where $\xi = kp + q$; k, p, and q are determined by (3.3).

Family 6. From (3.3), when $h_1 = h_3 = 0$, $h_0 = \frac{h_2m^2}{h_4(m^2+1)^2}$, $h_2 < 0$, and $h_4 > 0$, we obtain

$$\begin{aligned} H_6 &= -\frac{1}{2}\frac{\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)}{C_1C_3} \pm \sqrt{h_4}C_3C_1\sqrt{-\frac{h_2m^2}{h_4\,(m^2+1)}}\mathrm{sn}\left(\sqrt{-\frac{h_2}{m^2+1}}\xi\right) \\ G_6 &= \pm\frac{1}{4}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1h_2 + \frac{\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1h_2m^2\mathrm{sn}^2\left(\sqrt{-\frac{h_2}{m^2+1}}\xi\right)}{2(m^2+1)} \\ &\quad \pm\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1\sqrt{\frac{h_2m^2}{(m^2+1)^2} - \frac{h_2^2m^2\mathrm{sn}^2\left(\sqrt{-\frac{h_2}{m^2+1}}\xi\right)}{(m^2+1)} + \frac{h_2^2m^4\mathrm{sn}^4\left(\sqrt{-\frac{h_2}{m^2+1}}\xi\right)}{(m^2+1)^2}} \end{aligned} \tag{3.9}$$

where $\xi = kp + q$; k, p, and q are determined by (3.3).

Family 7. From (3.3), when $h_1 = h_3 = 0$, $h_4 = 1$, $h_2 = -(m^2+1)$, and $h_0 = m^2$, we obtain:

$$\begin{aligned} H_7 &= -\frac{\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)}{2C_3C_1} \pm C_3C_1\mathrm{ns}(\xi), \\ G_7 &= \pm\frac{1}{16}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1(-4m^2+4) - \frac{1}{2}C_1C_3\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)\mathrm{ns}^2(\xi) \\ &\quad \pm\frac{1}{2}C_1C_3\left(\frac{d}{\mathrm{d}y}F_2(y)\right)\sqrt{m^2+(-m^2+1)\mathrm{ns}^2(\xi)+\mathrm{ns}^4(\xi)}, \end{aligned} \tag{3.10}$$

where $\xi = kp + q$; k, p, and q are determined by (3.3).

Family 8. From (3.3), when $h_1 = h_3 = 0$, $h_4 = 1$, $h_2 = -(m^2+1)$, and $h_0 = m^2$, we obtain:

$$\begin{aligned} H_8 &= -\frac{\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)}{2C_3C_1} \pm C_3C_1\mathrm{dc}(\xi), \\ G_8 &= \pm\frac{1}{16}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1(-4m^2+4) - \frac{1}{2}C_1C_3\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)\mathrm{dc}^2(\xi) \\ &\quad \pm\frac{1}{2}C_1C_3\left(\frac{d}{\mathrm{d}y}F_2(y)\right)\sqrt{m^2+(-m^2+1)\mathrm{dc}^2(\xi)+\mathrm{dc}^4(\xi)}, \end{aligned} \tag{3.11}$$

where $\xi = kp+q$; k, p, and q are determined by (3.3).

Family 9. From (3.3), when $h_1 = h_3 = 0$, $h_4 = 1-m^2$, $h_2 = 2m^2-1$, and $h_0 = -m^2$, we obtain:

$$\begin{aligned} H_9 &= -\frac{\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)}{2C_1C_3} \pm \sqrt{1-m^2}C_3C_1\mathrm{nc}(\xi), \\ G_9 &= \pm\frac{1}{4}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1(2m^2-1) - \frac{1}{2}C_1C_3(1-m^2)\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)\mathrm{nc}^2(\xi) \\ &\quad \pm\frac{1}{2}\sqrt{1-m^2}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1\sqrt{-m^2+(2m^2-1)\mathrm{nc}^2(\xi)+(1-m^2)\mathrm{nc}^4(\xi)}, \end{aligned} \tag{3.12}$$

where $\xi = kp+q$; k, p, and q are determined by (3.3).

Family 10. From (3.3), when $h_1 = h_3 = 0$, $h_4 = m^2-1$, $h_2 = 2-m^2$, and $h_0 = -1$, we obtain:

$$\begin{aligned} H_{10} &= -\frac{\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)}{2C_1C_3} \pm \sqrt{m^2-1}C_3C_1\mathrm{nd}(\xi), \\ G_{10} &= \pm\frac{1}{4}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1(2-m^2) - \frac{1}{2}C_1C_3(m^2-1)\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)\mathrm{nd}^2(\xi) \\ &\quad \pm\frac{1}{2}\sqrt{m^2-1}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1\sqrt{-1+(2-m^2)\mathrm{nd}^2(\xi)+(m^2-1)\mathrm{nd}^4(\xi)}, \end{aligned} \tag{3.13}$$

where $\xi = kp+q$; k, p, and q are determined by (3.3).

Family 11. From (3.3), when $h_1 = h_3 = 0$, $h_4 = 1$, $h_2 = 2-m^2$, and $h_0 = 1-m^2$, we obtain:

$$\begin{aligned} H_{11} &= -\frac{\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)}{2C_3C_1} \pm C_3C_1\mathrm{cs}(\xi), \\ G_{11} &= \pm\frac{1}{16}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1(8-4m^2) - \frac{1}{2}C_1C_3\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)\mathrm{cs}^2(\xi) \\ &\quad \pm\frac{1}{2}C_1C_3\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)\sqrt{1-m^2+(2-m^2)\mathrm{cs}^2(\xi)+\mathrm{cs}^4(\xi)}, \end{aligned} \tag{3.14}$$

where $\xi = kp+q$; k, p, and q are determined by (3.3).

Family 12. From (3.3), when $h_1 = h_3 = 0$, $h_4 = 1-m^2$, $h_2 = 2-m^2$, and $h_0 = 1$, we obtain:

$$\begin{aligned} H_{12} &= -\frac{\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)}{2C_1C_3} \pm \sqrt{1-m^2}C_3C_1\mathrm{sc}(\xi), \\ G_{12} &= \pm\frac{1}{4}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1(2-m^2) - \frac{1}{2}C_1C_3(1-m^2)\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)\mathrm{sc}^2(\xi) \\ &\quad \pm\frac{1}{2}\sqrt{1-m^2}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1\sqrt{1+(2-m^2)\mathrm{sc}^2(\xi)+(1-m^2)\mathrm{sc}^4(\xi)}, \end{aligned} \tag{3.15}$$

where $\xi = kp+q$; k, p, and q are determined by (3.3).

Family 13. From (3.3), when $h_1 = h_3 = 0$, $h_4 = m^2(m^2-1)$, $h_2 = 2m^2-1$, and $h_0 = 1$, we obtain:

$$H_{13} = -\frac{\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)}{2C_1C_3} \pm \sqrt{m^2(m^2-1)}C_3C_1\mathrm{sd}(\xi),$$

$$\begin{aligned} G_{13} = &\pm\frac{1}{4}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1(2m^2-1) - \frac{1}{2}C_1C_3m^2(m^2-1)\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)\mathrm{sd}^2(\xi) \\ &\pm\frac{1}{2}\sqrt{m^2(m^2-1)}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1\sqrt{1+(2m^2-1)\mathrm{sd}^2(\xi)+m^2(m^2-1)\mathrm{sd}^4(\xi)}, \end{aligned} \tag{3.16}$$

where $\xi = kp+q$; k, p, and q are determined by (3.3).

Family 14. From (3.3), when $h_1 = h_3 = 0$, $h_4 = 1$, $h_2 = 2m^2-1$, and $h_0 = m^2(m^2-1)$, we obtain:

$$H_{14} = -\frac{\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)}{2C_3C_1} \pm C_3C_1\mathrm{sd}(\xi),$$

$$\begin{aligned} G_{14} = &\pm\frac{1}{16}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1(8m^2-4) - \frac{1}{2}C_1C_3\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)\mathrm{ds}^2(\xi) \\ &\pm\frac{1}{2}C_1C_3\left(\frac{d}{\mathrm{d}y}F_2(y)\right)\sqrt{m^2(m^2-1)+(2m^2-1)\mathrm{ds}^2(\xi)+\mathrm{ds}^4(\xi)}, \end{aligned} \tag{3.17}$$

where $\xi = kp+q$; k, p, and q are determined by (3.3).

Family 15. From (3.3), when $h_1 = h_3 = 0$, $h_4 = \frac{1}{4}$, $h_2 = \frac{1-2m^2}{2}$, and $h_0 = \frac{1}{4}$, we obtain:

$$H_{15} = -\frac{\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)}{2C_3C_1} \pm \frac{1}{2}C_3C_1(\mathrm{ns}(\xi)\pm\mathrm{cs}(\xi)),$$

$$\begin{aligned} G_{15} = &\pm\frac{1}{4}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1\left(\frac{1}{2}-m^2\right) - \frac{1}{8}C_1C_3\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)(\mathrm{ns}(\xi)\pm\mathrm{cs}(\xi))^2 \\ &\pm\frac{1}{8}C_1C_3\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)\sqrt{1+4\left(\frac{1}{2}-m^2\right)(\mathrm{ns}(\xi)\pm\mathrm{cs}(\xi))^2+(\mathrm{ns}(\xi)+\mathrm{cs}(\xi))^4}, \end{aligned} \tag{3.18}$$

where $\xi = kp+q$; k, p, and q are determined by (3.3).

Family 16. From (3.3), when $h_1 = h_3 = 0$, $h_4 = \frac{1-m^2}{4}$, $h_2 = \frac{1+m^2}{2}$, and $h_0 = \frac{1-m^2}{4}$, we obtain:

$$H_{16} = -\frac{\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)}{2C_1C_3} \pm \frac{1}{2}\sqrt{1-m^2}C_3C_1(\mathrm{nc}(\xi)\pm\mathrm{sc}(\xi)),$$

$$\begin{aligned} G_{16} = &\pm\frac{1}{8}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1(1+m^2) - \frac{1}{8}C_1C_3(1-m^2)\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)(\mathrm{nc}(\xi)\pm\mathrm{sc}(\xi))^2 \\ &\pm\frac{1}{8}\sqrt{1-m^2}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1\sqrt{1-m^2+2(1+m^2)(\mathrm{nc}(\xi)\pm\mathrm{sc}(\xi))^2+(1-m^2)(\mathrm{nc}(\xi)\pm\mathrm{sc}(\xi))^4}, \end{aligned} \tag{3.19}$$

where $\xi = kp+q$; k, p, and q are determined by (3.3).

Family 17. From (3.3), when $h_1 = h_3 = 0$, $h_4 = \frac{1}{4}$, $h_2 = \frac{m^2-2}{2}$, and $h_0 = \frac{m^2}{4}$, we obtain:

$$H_{17} = -\frac{\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)}{2C_3C_1} \pm \frac{1}{2}C_3C_1(\mathrm{ns}(\xi) \pm \mathrm{ds}(\xi)),$$

$$G_{17} = \pm\frac{1}{8}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1(m^2-2) - \frac{1}{8}C_1C_3\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)(\mathrm{ns}(\xi) \pm \mathrm{ds}(\xi))^2 \pm \frac{1}{8}C_1C_3\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)\sqrt{m^2 + 2(m^2-2)(\mathrm{ns}(\xi) \pm \mathrm{ds}(\xi))^2 + (\mathrm{ns}(\xi) \pm \mathrm{ds}(\xi))^4}, \quad (3.20)$$

where $\xi = kp + q$; k, p, and q are determined by (3.3).

Family 18. From (3.3), when $h_1 = h_3 = 0$, $h_4 = \frac{m^2}{4}$, $h_2 = \frac{m^2-2}{2}$, $h_0 = \frac{m^2}{4}$, we obtain:

$$H_{18} = -\frac{\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)}{2C_1C_3} \pm \frac{1}{2}\sqrt{m^2}C_3C_1(\mathrm{sn}(\xi) \pm \mathrm{icn}(\xi)),$$

$$G_{18} = \pm\frac{1}{8}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1(m^2-2) - \frac{1}{8}C_1C_3m^2\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)(\mathrm{sn}(\xi) \pm \mathrm{icn}(\xi))^2 \pm \frac{1}{8}\sqrt{m^2}\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)C_3C_1\sqrt{m^2 + 2(m^2-2)(\mathrm{sn}(\xi) \pm \mathrm{icn}(\xi))^2 + m^2(\mathrm{sn}(\xi) \pm \mathrm{icn}(\xi))^4}, \quad (3.21)$$

where $\xi = kp + q$; k, p, and q are determined by (3.3).

Remark: It is necessary to point out that C_1 and C_3 are free constants and that h_1, h_2, h_3, and h_4 $(h_4 \neq 0)$ are arbitrary constants. Then, due to the arbitrariness of h_1, h_2, h_3, and h_4 $(h_4 \neq 0)$, it is possible to give different values to obtain many families of solutions.

In order to better understand the properties of the solutions obtained here, eight figures (Figs. 1 – 4) are drawn to illustrate soliton-like solutions.

4. Summary and Conclusions

By using of a new and more general ansatz and with the aid of a symbolic computation system Maple, we extended the unified algebraic method [17, 18] proposed by Fan and the improved extended tanh method [20] by Yomba to uniformly construct a series of a series of soliton-like solutions and double periodic solutions for nonlinear partial differential equations. We apply the generalized method to solve a $(2+1)$-dimensional Broer-Kaup-Kupershmidt equation and successfully construct new and more general solutions including a series of nontravelling wave and coefficient functions' soliton-like solutions, double periodic solutions and triangular solutions. The method can easily be extended to other NPDEs and is sufficient to seek more new formal solutions of NPDEs.

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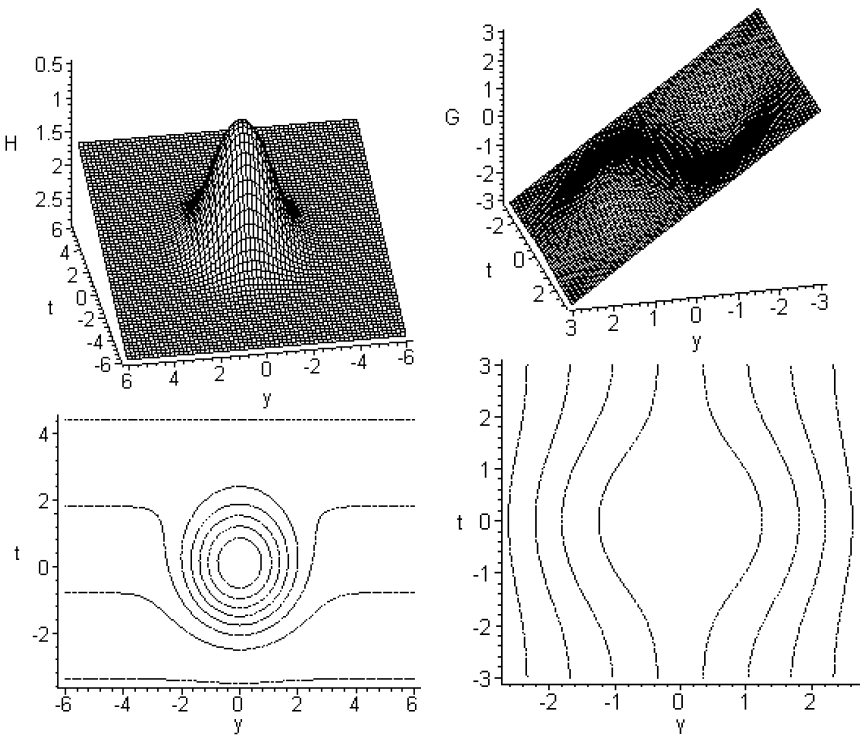

Fig. 1. The soliton-like solution H_2 and G_2, where $C_1 = 10, C_2 = C_3 = 1$, $h_2 = -0.1$, $F_1(t) = t^2$, $F_2(y) = y^2$, and $x = 0$.

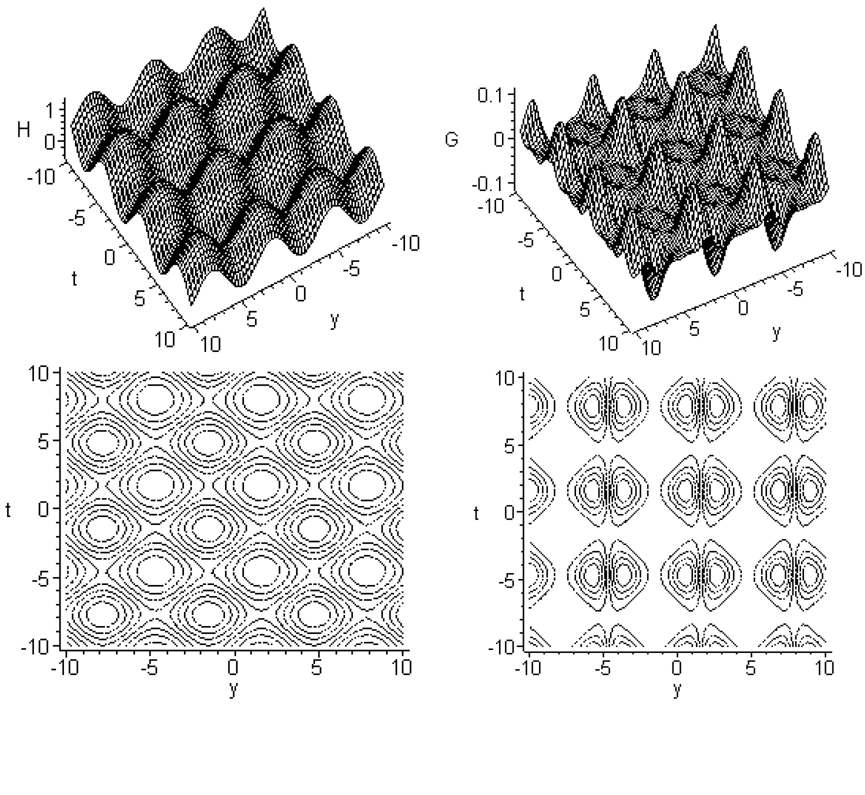

Fig. 2. The soliton-like solution H_2 and G_2, where $C_1 = 10$, $C_2 = C_3 = 1$, $h_2 = -0.1$, $F_1(t) = \sin(t)$, $F_2(y) = \sin(y)$, and $x = 0$.

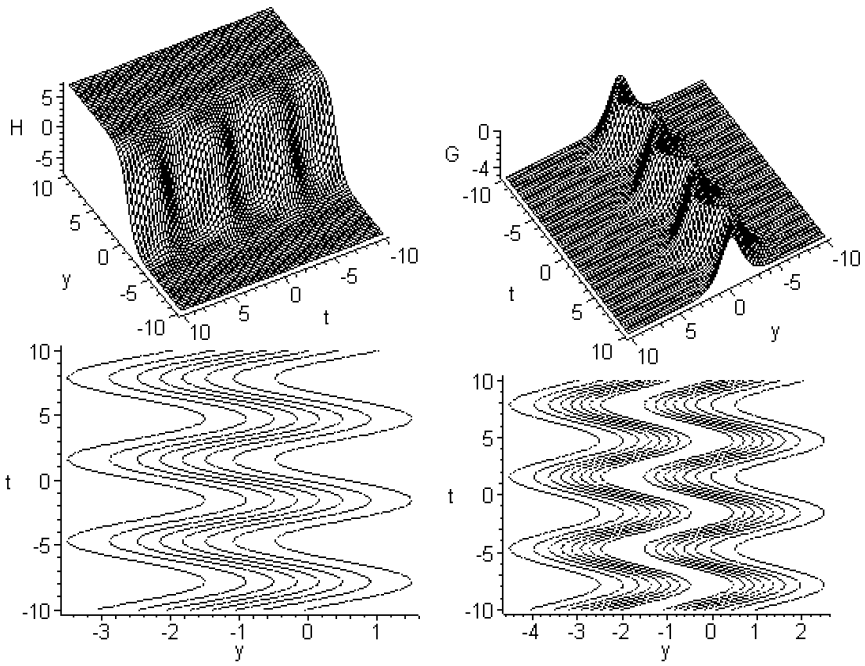

Fig. 3. The soliton-like solution H_2 and G_2, where $C_1 = 10$, $C_2 = C_3 = 1$, $h_2 = -0.1$, $F_1(t) = \sin(t)$, $F_2(y) = y$, and $x = 0$.

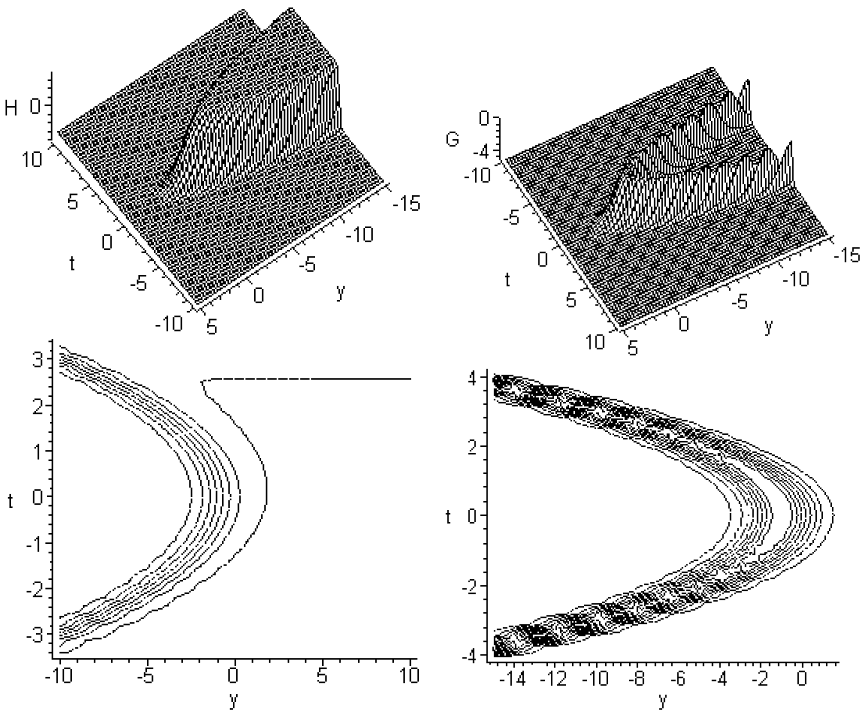

Fig. 4. The soliton-like solution H_2 and G_2, where $C_1 = 10, C_2 = C_3 = 1$, $h_2 = -0.1$, $F_1(t) = t^2$, $F_2(y) = y$, and $x = 0$.

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